North Sydney Girls High School

2024 **HSC TRIAL EXAMINATION**

Mathematics Extension 2

General Instructions

- Reading Time – 10 minutes
- Working Time – 3 hours
- Write using black pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- For questions in Section II, show relevant mathematical reasoning and/or calculations

Total marks: 100

Section I – 10 marks (pages 2 – 5)

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 6 – 15)

- Attempt Questions 11 – 16
- Allow about 2 hours and 45 minutes for this section

NAME:________________________ **TEACHER:**________________________

STUDENT NUMBER:

Question	1–10	11	12	13	14	15	16	Total
Mark	/10	/14	/14	/16	/17	/14	/15	/100

Section I

10 marks
Attempt Questions 1-10
Allow about 15 minutes for this section

Use the multiple choice answer sheet for Questions 1−10.

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1 What is the length of the vector $-2\underset{\sim}{i}-6\underset{\sim}{j}+9\underset{\sim}{k}$?

A. 1

B. 11

C. 17

D. 121

2 Which of the following expressions is equivalent to $\int \ln\left(x^2+1\right)dx$?

A. $x\ln\left(x^2+1\right)-2x+2\tan^{-1}x+c$

B. $x\ln\left(x^2+1\right)-\ln\left(x^2+1\right)+c$

C. $\ln\left(x^2+1\right)-2x+2\tan^{-1}x+c$

D. $\ln\left(x^2+1\right)-x\ln\left(x^2+1\right)+c$

3 The polynomial $P(z)$ has real coefficients and $z=-2+i$ is a zero of $P(z)$.

Which quadratic polynomial must be a factor of $P(z)$?

A. z^2+4z+5

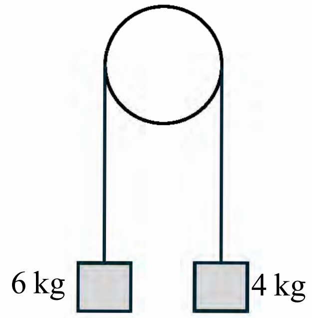
B. z^2+4z+3

C. z^2-4z+5

D. z^2-4z+3

4 A light inextensible string passes over a frictionless pulley.

Masses of 4 kg and 6 kg are attached to the ends of the string as shown, and the acceleration due to gravity is $10\ \mathrm{ms}^{-2}$.



If the system starts at rest, how fast in m/s will the masses be moving 2 seconds later?

A. 60

B. 6

C. 4

D. 2

5 A line in 3D space has equation $\underset{\sim}{r} = \begin{bmatrix} 6 \\ -4 \\ 2 \end{bmatrix} + \lambda \begin{bmatrix} -3 \\ 5 \\ 4 \end{bmatrix}$, where λ is a real constant.

Which of the following statements is true?

A. The line passes through the origin.

B. The point $(-3, 5, 4)$ lies on the line.

C. The point $(9, -9, -2)$ lies on the line.

D. The line points in the direction of the vector $3\underset{\sim}{i} - 2\underset{\sim}{j} + \underset{\sim}{k}$.

6 The complex number z satisfies $|z| - z = 4(1-2i)$.

What is the value of $|z|^2$?

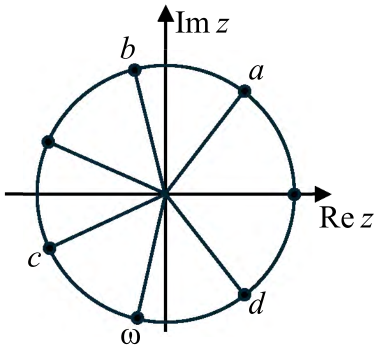
A. 80

B. 100

C. 180

D. 400

7 The seven roots of the equation $z^7 = 1$, including the complex numbers a, b, c, d and ω, are shown in the following diagram.



Which of the following 7^{th} roots of 1 is also a cube root of ω?

A. a

B. b

C. c

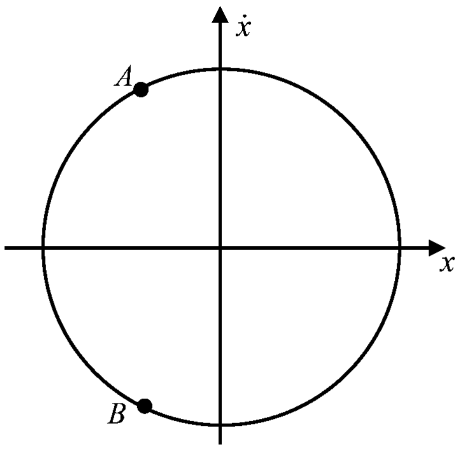
D. d

8 A particle undergoes simple harmonic motion according to the equation

$$x = -2\cos\left(t - \frac{\pi}{3}\right)$$

where x measures the displacement of the particle from an origin and t is the time since the particle began moving.

Below is a graph of velocity against displacement for this particle.



A point P on this graph moves so as to represent the changing position and velocity of the particle.

Which statement about P could be correct?

A. P starts at A and moves clockwise

B. P starts at A and moves anticlockwise

C. P starts at B and moves clockwise

D. P starts at B and moves anticlockwise

9 Which of the following statements is FALSE?

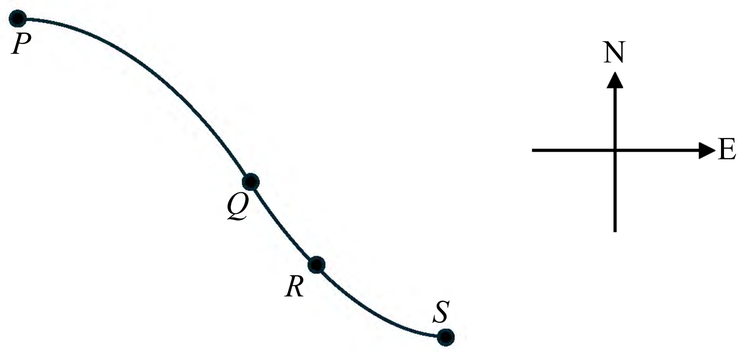
A. $\forall x \in \mathbb{R},\ x^2 \geq 0$

B. $\forall n \in \mathbb{Z},\ \exists m \in \mathbb{Z},\ m = n + 5$

C. $\exists a \in \mathbb{R},\ \forall x \in \mathbb{R},\ ax = x$

D. $\exists m \in \mathbb{Z},\ \forall n \in \mathbb{Z},\ m = n + 5$

10 The diagram shows part of the trajectory of a particle moving in a two-dimensional plane.



P and S are stationary points of the curve, and Q is the only point of inflection.

The particle is speeding up as it passes through P, and continues to speed up until it reaches R, then slows down until it reaches S without ever reaching a speed of zero.

Where is it possible for the particle's acceleration vector to be pointing due East?

A. Between P and Q only

B. Between Q and R only

C. Between R and S only

D. Between P and Q and also between Q and R

Section II

90 marks
Attempt Questions 11−16
Allow about 2 hours and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

For questions in Section II, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (14 marks**)** Use a SEPARATE writing booklet

(a) Solve the equation $z^2 - 3iz - 2 = 0$, giving your answers in exponential form. **2**

(b) If $z = 4e^{i\frac{\pi}{3}}$ and $\omega = 2i$, evaluate $\frac{z}{\omega}$ in the form $a + ib$. **2**

(c) Find a vector equation for the line which passes through the points **2**

$$A(1, 3, -2) \text{ and } B(0, 1, 1).$$

(d) Consider the points $A(-2, 3, 1)$ and $B(2, 7, -3)$ on a 3D Cartesian plane.

(i) If O is the origin, find the vector $\frac{\overrightarrow{OA} + \overrightarrow{OB}}{2}$ in component form. **1**

(ii) A and B are the endpoints of the diameter of a sphere. **2**

Find a vector equation for the sphere.

(e) Prove that $\frac{3}{4}a^4 + \frac{1}{3}b^2 \geq a^2 b \quad \forall a, b \in \mathbb{R}$. **2**

(Do NOT assume the AM-GM inequality.)

(f) Use the method of partial fractions to find $\int \frac{6 - 5x}{4 - x^2}\, dx$. **3**

Question 12 (14 marks) Use a SEPARATE writing booklet

(a) Use an appropriate substitution to find $\int \frac{\sqrt{25x^2-4}}{x}\,dx$. **3**

(b) A particle moves in a straight line with initial displacement $x=0$.

The velocity of the particle is given by $v=2e^{-\frac{x}{2}}(x+1)^2$, where v is in metres per second.

(i) Show that the acceleration of the particle is given by $a=2e^{-x}(x+1)^3(3-x)$. **2**

(ii) Hence find the displacement for which the maximum velocity of the particle will occur. Justify your answer. **2**

(c) Shade the region in the complex plane containing all points which satisfy **3**

$$|z+4i|\le 3|z|.$$

Show all working.

(d) (i) Use the result $e^{i\theta}=\cos\theta+i\sin\theta$ to show that $e^{ni\theta}+e^{-ni\theta}=2\cos n\theta$ for $n\in\mathbb{R}$. **1**

(ii) Hence find $\int \cos^4\theta\,d\theta$. **3**

Question 13 (16 marks) Use a SEPARATE writing booklet

(a) The interval AB subtends an angle of $60°$ at the origin of a 3D number plane, where A and B have coordinates $(2, 0, -2)$ and $(4, 1, 2m)$ respectively. **3**

Find the possible value(s) of m.

(b) Use integration by parts to find $\int e^{-x} \cos 2x \, dx$. **3**

(c) Consider the statement: If a^3 is even then a is even.

(i) Write down the contrapositive of this statement. **1**

(ii) Hence prove that the statement is true. **2**

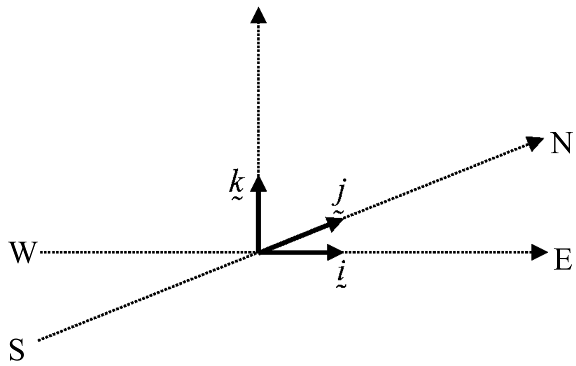
You are given integers p and q which satisfy $(p-q)^3 + p^3 = (p+q)^3$.

Rearranging this relation leads to $p^3 = 2\left(3p^2q + q^3\right)$. (Do NOT show this)

(iii) Show that q is even. **3**

Question 13 continues on page 9

(d) Relative to a fixed origin O, the unit vectors $\underset{\sim}{i}$, $\underset{\sim}{j}$ and $\underset{\sim}{k}$ point East, North and vertically upwards, respectively.



Two missiles A and B are fired simultaneously and have paths described as follows:

$$A(3t-19,\ 2t-14,\ 28-t) \quad \text{and} \quad B(2t-11,\ 10-t,\ 2t+4)$$

where t represents the time in minutes after being fired, and all distances are in kilometres.

(i) It is desired to find where the paths of A and B cross. **1**

Explain briefly *in terms of the context of this question* why it is necessary to use different pronumerals for the time parameters of A and B.

(ii) Show that A and B will collide, stating the coordinates of the point of impact P. **3**

End of Question 13

Question 14 (17 marks) Use a SEPARATE writing booklet

(a) A particle moves in simple harmonic motion in a straight line.

The velocity v m/s of the particle at a displacement x metres from the origin, is given by

$$v^2 = 32 + 8x - 4x^2 .$$

(i) Find the amplitude and period of the motion. **2**

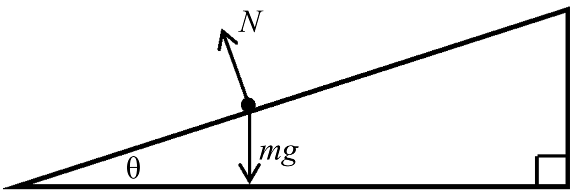
(ii) The particle is first observed at a point 2.5 metres to the right of the origin moving to the right. Find a possible equation for the displacement of the particle at time t seconds after it was first observed. **2**

(iii) At what time will the particle first be seen passing though the origin and increasing in speed? **3**

(b) A slope is inclined at an angle θ to the horizontal, where $\tan\theta = \frac{4}{3}$.

An object of mass m kg is projected up the slope with a speed of V m/s and comes instantaneously to rest after travelling 6 metres.

A frictional force F opposes the motion and the coefficient of friction is 0.5.



(i) By resolving the forces perpendicular to the plane, show that the frictional force is $F = \frac{3mg}{10}$. **2**

(ii) Resolve the forces parallel to the plane and find the initial velocity V correct to three significant figures. Use $g = 9.8\ \text{ms}^{-2}$. **3**

Question 14 continues on page 11

(c) The integers 1 to 30 are to be arranged randomly around a circle so that each integer occurs exactly once. Consider the statement:

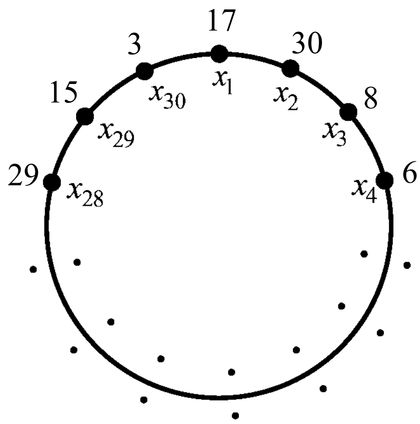
P: *There exists a set of three neighbouring integers around the circle which sum to at least 45.*

(i) Without using the word "exist" or an equivalent, write down the negation of this statement. **1**

An arbitrarily chosen integer on the circle is labelled x_1, and the other integers are labelled consecutively around the circle $x_2, x_3, x_4, ..., x_{30}$.

The diagram below shows an example for the partial arrangement

..., 29, 15, 3, 17, 30, 8, 6, ...



(ii) Assume that the statement P is FALSE. **1**

Hence write down an inequality (not specific to the example above) involving x_1, x_2 and x_3, and another inequality involving x_{30}, x_1 and x_2.

(iii) Prove by contradiction that the statement P is true for any random arrangement. **3**

(A proof by the pigeonhole principle will receive no credit.)

End of Question 14

Question 15 (14 marks) Use a SEPARATE writing booklet

(a) (i) Show that $\int_a^b f(a+b-x)\,dx = \int_a^b f(x)\,dx$ for real constants *a* and *b*. **1**

(ii) Hence evaluate $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{dx}{1+\sqrt{\tan x}}$. **3**

(b) Let $P(z) = az^2 + ibz + c$, where *a*, *b* and *c* are real constants. **2**

Let ω be a complex root of $P(z)=0$.

Use properties of complex conjugates to show that $-\overline{\omega}$ is also a root of $P(z)=0$.

(c) (i) By taking the dot product of two well-chosen vectors, prove for $a, b, c \in \mathbb{R}$ that: **2**

If $a+b+c=12$ then $a^2+b^2+c^2 \geq 48$.

(ii) Provide a geometric reason with reference to your chosen vectors why equality occurs only when $a=b=c=4$. **1**

(d) Let $I_k = \int_0^1 x^k (1-x)^{n-k}\,dx$ for integers *n* and *k*, where $0 \leq k \leq n$.

(i) Show that $I_k = \frac{k}{n-k+1} I_{k-1}$. **2**

(ii) Hence show that $\int_0^1 \binom{n}{k} x^k (1-x)^{n-k}\,dx = \frac{1}{n+1}$. **3**

Question 16 (15 marks) Use a SEPARATE writing booklet

(a) In the country of Discretia, the currency consists of \$3 and \$8 coins. **3**

Use mathematical induction to prove that any whole dollar amount of \$14 or more can be achieved by taking a combination of \$3 and \$8 coins.

Hint: You may wish to consider two cases in the inductive step −

no \$8 coins and at least one \$8 coin.

(b) (i) Show that $\frac{2k-1}{2k} < \frac{2k}{2k+1}$ for all positive integers k. **1**

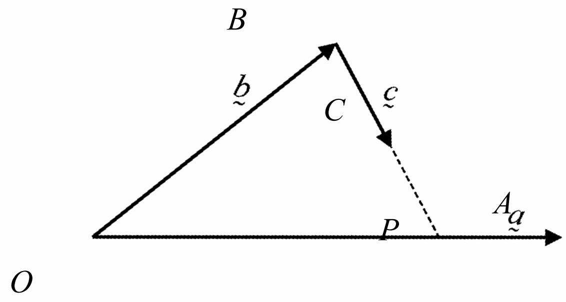
(ii) By considering $p = \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \ldots \cdot \frac{2n-1}{2n}$ and $q = \frac{2}{3} \cdot \frac{4}{5} \cdot \frac{6}{7} \cdot \ldots \cdot \frac{2n}{2n+1}$, **3**

show that $\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdot \ldots \cdot \frac{2n-1}{2n} < \frac{1}{\sqrt{2n}}$ for all positive integers n.

Question 16 continues on page 14

(c) (i) Any vector $\underset{\sim}{c}$ in two dimensions can be expressed uniquely in the form $\underset{\sim}{c} = \lambda\underset{\sim}{a} + \mu\underset{\sim}{b}$ **2**
for non-zero and non-parallel vectors $\underset{\sim}{a}$ and $\underset{\sim}{b}$, where λ and μ are real scalars.

The following diagram shows three non-zero vectors $\overrightarrow{OA} = \underset{\sim}{a}$, $\overrightarrow{OB} = \underset{\sim}{b}$ and $\overrightarrow{BC} = \underset{\sim}{c}$ such that no two vectors are parallel and all are non-zero.



BC produced meets OA at P, such that $\overrightarrow{OP} = \alpha\overrightarrow{OA}$ and $\overrightarrow{BP} = \beta\overrightarrow{BC}$,
where $0 < \alpha < 1 < \beta$.

By equating two expressions for $\overrightarrow{OP}$, show that

$$\alpha = -\frac{\lambda}{\mu}$$

when $\underset{\sim}{c}$ is expressed in the form $\lambda\underset{\sim}{a} + \mu\underset{\sim}{b}$.

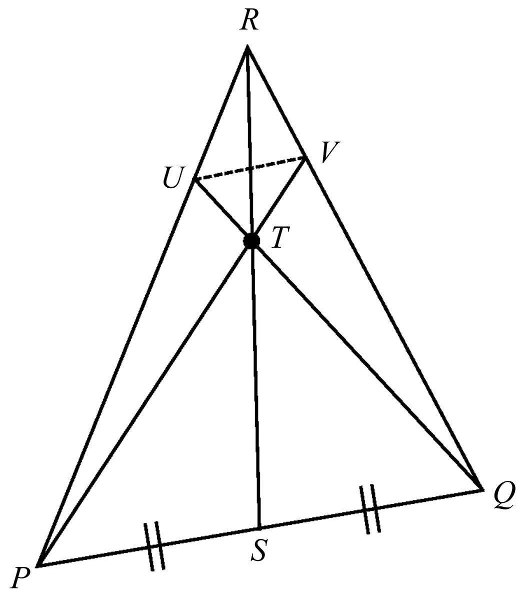
Question 16 continues on page 15

Question 16 (continued)

(ii) In triangle PQR, the median is drawn from the midpoint S of PQ to the opposite vertex R. **3**

The point T is chosen arbitrarily on RS.

QT is produced to U on PR, and PT is produced to V on QR.



Let $\overrightarrow{RT} = \phi \overrightarrow{RS}$.

Express $\overrightarrow{QT}$ and $\overrightarrow{PT}$ in terms of $\overrightarrow{RP}$ and $\overrightarrow{RQ}$ and hence use part (i) to prove that UV is parallel to PQ.

.

(d) Let $f_0(x) = \dfrac{1}{1-x}$, and define $f_k(x) = f_0\left(f_{k-1}(x)\right)$. **3**

Show that if $m = 3p + a$, $n = 3q + b$, $p, q \in \{0,1,2,\ldots\}$, $a, b \in \{0,1,2\}$, $a \neq b$,

then the only solutions to $f_m(x) = f_n(x)$ are $x = e^{\pm i\frac{\pi}{3}}$.

End of paper

Multiple Choice

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1	B	2	A	3	A	4	C	5	C
6	B	7	C	8	C	9	D	10	B

① $|z|^2 = 2^2 + 6^2 + 9^2 = 121$

$|z| = 11$

② $\int \ln(x^2+1)\, d(x) = x\ln(x^2+1) - \int x \cdot d[\ln(x^2+1)]$

$= x\ln(x^2+1) - \int x \cdot \frac{2x}{x^2+1} dx$

$= x\ln(x^2+1) - 2\int \frac{x^2}{x^2+1} dx$

$= x\ln(x^2+1) - 2\int \left(1 - \frac{1}{x^2+1}\right) dx$

$= x\ln(x^2+1) - 2x + 2\tan^{-1} x + c$

③ If $z = -2+i$ is a zero, so is $\bar{z} = -2-i$

$z + \bar{z} = -4$

$z\bar{z} = 5$

$\therefore z^2 + 4z + 5$ is a factor

④ $(6+4)\ddot{x} = 6g - 4g$

$\ddot{x} = 2$

$v = 2 \times 2 = 4$

⑤ (A) $x = 0 \Rightarrow \lambda = 2$, $\lambda = 2 \Rightarrow y = 6$ ✗

(B) $x = -3 \Rightarrow \lambda = 3$, $\lambda = 3 \Rightarrow y = 11$ ✗

(C) $x = 9 \Rightarrow \lambda = -1$, $\lambda = -1 \Rightarrow y = -9, z = -2$ ✓

(D) $(-3, 5, 4)$ is the direction vector, not $(6, -4, 2)$ ✗

⑥ $\sqrt{x^2+y^2} - x - iy = 4 - 8i$

$y = -8$

$\sqrt{x^2+64} - x = 4$

$\sqrt{x^2+64} = x + 4$

$\cancel{x^2} + 64 = \cancel{x^2} + 8x + 16$

$8x = 48$

$x = 6$

$z = 6 - 8i$

$|z|^2 = 100$

(7) $w = \text{cis}\,\frac{10\pi}{7}$

$w^{1/3} = \text{cis}\,\frac{10\pi}{21},\ \text{cis}\left(\frac{10\pi}{21} + \frac{2\pi}{3}\right),\ \text{cis}\left(\frac{10\pi}{21} - \frac{2\pi}{3}\right)$

$= \text{cis}\,\frac{10\pi}{21},\ \text{cis}\,\frac{8\pi}{7},\ \text{cis}\left(-\frac{4\pi}{21}\right)$

$\text{cis}\,\frac{8\pi}{7}$ is C

(8) $x = -2\cos\left(t - \frac{\pi}{3}\right) \Rightarrow x_0 = -1 < 0$

$\dot{x} = 2\sin\left(t - \frac{\pi}{3}\right) \Rightarrow \dot{x}_0 = -\sqrt{3} < 0$

$\therefore$ starts at B

as $\dot{x}_0 < 0$, x will decrease (become more negative) $\Rightarrow$ clockwise

(9) (A) clearly true ✓

(B) translating: For every integer n I can find an m 5 larger than it ✓

(C) translating: I can find an a [a = 0] such that every number times a is zero ✓

(D) translating: I can find an integer m such that every integer is 5 more than m ✗

(10) Consider the acceleration vector to have two components:

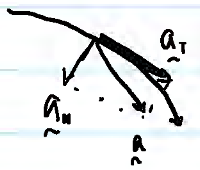
(1) tangential to the trajectory, $\underset{\sim}{a}_T$

(2) normal to the trajectory, $\underset{\sim}{a}_N$

$\underset{\sim}{a}_N$ indicates the direction the object is turning

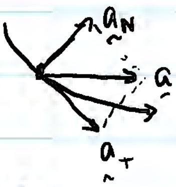
$\underset{\sim}{a}_T$ indicates whether the object is speeding up or slowing

(I) PQ:



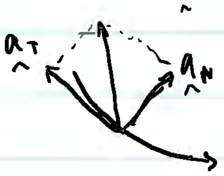
turning right and speeding up

(II) QR:



turning left and speeding up

(III) RS:



turning left and slowing

Only (III) can point due east

Question 11

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(a) $z^2 = 3iz + 2$

$z^2 - 3iz - 2 = 0$

$z = \left(3i \pm \sqrt{(3i)^2 + 8}\right)/2$

$= \left(3i \pm \sqrt{-1}\right)/2$

$= (3i \pm i)/2$

$z = i, 2i$

$\underline{z = e^{i\frac{\pi}{2}}, 2e^{i\frac{\pi}{2}}}$ (1)

(b) $z = 4e^{i\frac{\pi}{3}}$, $w = 2e^{i\frac{\pi}{2}}$

$\frac{z}{w} = 2e^{-i\frac{\pi}{6}}$

$= 2\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)$

$= \underline{\sqrt{3} - i}$

OR $z = 2 + 2\sqrt{3}i$, $w = 2i$

$\frac{z}{w} = \frac{1+\sqrt{3}i}{i} \times \frac{i}{i}$

$= \frac{-\sqrt{3} + i}{-1}$

$= \underline{\sqrt{3} - i}$ (2)

(c) direction vector: $\underset{\sim}{d} = \overrightarrow{AB}$

$= \overrightarrow{OB} - \overrightarrow{OA}$

$= (\underset{\sim}{j} + \underset{\sim}{k}) - (\underset{\sim}{i} + 3\underset{\sim}{j} - 2\underset{\sim}{k})$

$= -\underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k}$

$\therefore \underset{\sim}{r} = \underline{(\underset{\sim}{i} + 3\underset{\sim}{j} - 2\underset{\sim}{k}) + \lambda(-\underset{\sim}{i} - 2\underset{\sim}{j} + 3\underset{\sim}{k})}$ OR $\underset{\sim}{r} = (1-\lambda)\underset{\sim}{i} + (3-2\lambda)\underset{\sim}{j} + (3\lambda - 2)\underset{\sim}{k}$ (2)

(d) (i) $\frac{\overrightarrow{OA} + \overrightarrow{OB}}{2} = \underline{5\underset{\sim}{j} - \underset{\sim}{k}}$ (1)

(ii) From (i), centre is $(0, 5, -1)$

radius is $\frac{1}{2}\left|\overrightarrow{AB}\right| = \frac{1}{2}\sqrt{(2+2)^2 + (7-3)^2 + (-3-1)^2} = 2\sqrt{3}$

$\therefore \left|\underset{\sim}{r} - (5\underset{\sim}{j} - \underset{\sim}{k})\right| = 2\sqrt{3}$ (2)

(e) $\frac{3}{4}a^4 + \frac{1}{3}b^2 - a^2 b = \frac{1}{12}\left(9a^4 + 4b^2 - 12a^2 b\right)$

$= \frac{1}{12}\left(3a^2 - 2b\right)^2$

≥ 0 (perfect square)

$\therefore \frac{3}{4}a^4 + \frac{1}{3}b^2 \geq a^2 b$ (2)

f) let $\frac{6-5x}{4-x^2} = \frac{A}{2+x} + \frac{B}{2-x}$

$6-5x = A(2-x) + B(2+x)$

$(x=2)$: $-4 = 4B \Rightarrow B = -1$

$(x=-2)$: $16 = 4A \Rightarrow A = 4$

$\therefore \int \frac{6-5x}{4-x^2}\,dx = \int \frac{4}{2+x}\,dx - \int \frac{1}{2-x}\,dx$

$= \underline{4\ln|2+x| + \ln|2-x| + c}$ (2)

$= \ln\left|(2-x)(2+x)^4\right| + c$

Question 12

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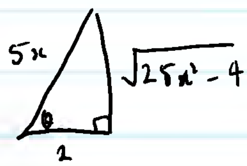
a) $\displaystyle\int \frac{\sqrt{25x^2-4}}{x}\,dx$

let $x = \frac{2}{5}\sec\theta \Rightarrow \sec\theta = \frac{5x}{2}$

$dx = \frac{2}{5}\sec\theta\tan\theta\,d\theta$

$= \displaystyle\int \frac{\sqrt{4\sec^2\theta-4}}{\cancel{\frac{2}{5}}\sec\theta}\cdot\cancel{\frac{2}{5}}\sec\theta\tan\theta\,d\theta$



$\Rightarrow \tan\theta = \frac{\sqrt{25x^2-4}}{2}$

$= \displaystyle\int \frac{2\tan\theta}{\cancel{\sec\theta}}\cdot\cancel{\sec\theta}\tan\theta\,d\theta$

$\sec\theta = \frac{5x}{2} \Rightarrow \cos\theta = \frac{2}{5x}$

$\Rightarrow \theta = \cos^{-1}\left(\frac{2}{5x}\right)$

$= \displaystyle\int 2\tan^2\theta\,d\theta$

$= \displaystyle\int (2\sec^2\theta - 2)\,d\theta$

$= 2\tan\theta - 2\theta + c$

$= \sqrt{25x^2-4} - 2\cos^{-1}\left(\frac{2}{5x}\right) + c$ $\left[\text{or } -2\sec^{-1}\left(\frac{5x}{2}\right)\right]$ (3)

See Alternate solution at end

(b) (i) $v = 2e^{-\frac{x}{2}}(x+1)^2$

$a = \frac{1}{2}\cdot\frac{d}{dx}(v^2)$

$= \frac{1}{2}\cdot\frac{d}{dx}\,4e^{-x}(x+1)^4$

$= 2\cdot\frac{d}{dx}\,e^{-x}(x+1)^4$

$= 2\left[e^{-x}\cdot 4(x+1)^3 + (x+1)^4\cdot -e^{-x}\right]$

$= 2e^{-x}(x+1)^3\left[4-(x+1)\right]$

$= 2e^{-x}(x+1)^3(3-x)$ Show (2)

(ii) Max velocity when $a=0 \Rightarrow x=-1$ or $x=3$

Initially $x=0 \Rightarrow v=2>0$, $a=6>0$

$\therefore$ moving to right and speeding up

$\therefore x=3$ is where a "first" equals zero

As $v>0\ \forall x$, it will never reach $x=-1$

$\therefore x=3$

As $a<0$ for $x>3$, particle slows after passing $x=3$

$\therefore$ max velocity when $x=3$ [2]

(c) $|z+4i| \leq 3|z|$

let $z = x+iy \Rightarrow$ $|x+iy+4i| \leq 3|x+iy|$

$|x+i(y+4)| \leq 3|x+iy|$

$|x+i(y+4)|^2 \leq 9|x+iy|^2$

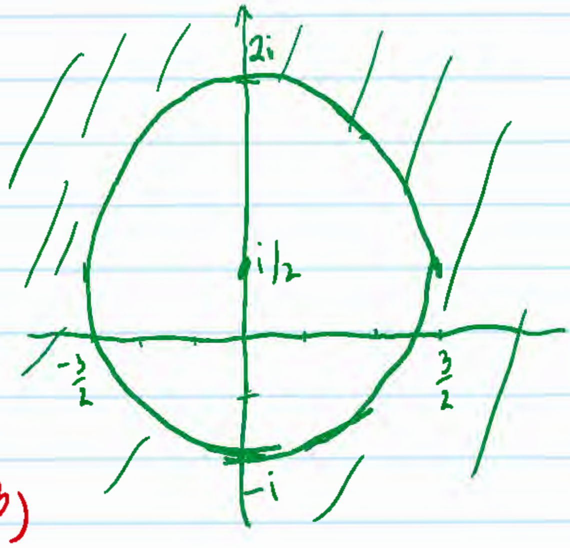
$x^2+(y+4)^2 \leq 9(x^2+y^2)$

$x^2+y^2+8y+16 \leq 9x^2+9y^2$

$8x^2+8y^2-8y \geq 16$

$x^2+y^2-y+\frac{1}{4} \geq 2+\frac{1}{4}$

$x^2+(y-\frac{1}{2})^2 \geq (\frac{3}{2})^2$ (3)



(d) (i) $e^{i\theta} = \cos\theta + i\sin\theta$

$e^{ni\theta}+e^{-ni\theta} = (\cos n\theta + i\sin n\theta) + (\cos(-n\theta) + i\sin(-n\theta))$ (de Moivre's theorem)

$= \cos n\theta + i\sin n\theta + \cos n\theta - i\sin n\theta$

$= 2\cos n\theta$ Show (1)

(ii) $(e^{i\theta}+e^{-i\theta})^4 = e^{4i\theta} + 4e^{2i\theta} + 6 + 4e^{-2i\theta} + e^{-4i\theta}$

$(2\cos\theta)^4 = (e^{4i\theta}+e^{-4i\theta}) + 4(e^{2i\theta}+e^{-2i\theta}) + 6$

$16\cos^4\theta = 2\cos 4\theta + 8\cos 2\theta + 6$

$\cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$

$\int \cos^4\theta \, d\theta = \frac{1}{32}\sin 4\theta + \frac{1}{4}\sin 2\theta + \frac{3\theta}{8} + c$ (3)

Question 13

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(a) $\vec{OA} = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}$, $\vec{OB} = \begin{pmatrix} 4 \\ 1 \\ 2m \end{pmatrix}$

$\vec{OA} \cdot \vec{OB} = |\vec{OA}| . |\vec{OB}| \cdot \cos 60^\circ$

$8 - 4m = \cancel{2}\sqrt{2} \cdot \sqrt{17 + 4m^2} \cdot \frac{1}{\cancel{2}}$

$(8 - 4m)^2 = 2(17 + 4m^2)$

$64 - 64m + 16m^2 = 34 + 8m^2$

$8m^2 - 64m + 30 = 0$

$4m^2 - 32m + 15 = 0$

$(2m - 1)(2m - 15) = 0$

$m = \frac{1}{2}, \frac{15}{2}$

But if $m = \frac{15}{2}$ then $\vec{OA} \cdot \vec{OB} < 0$, but the angle between $\vec{OA}$ & $\vec{OB}$ is acute

$\therefore \underline{m = \frac{1}{2}}$ (3)

(b) $\int e^{-x} \cos 2x \, dx = \int e^{-x} \cdot d\left(\frac{1}{2} \sin 2x\right)$

$= \frac{1}{2} e^{-x} \sin 2x - \frac{1}{2} \int \sin 2x \cdot d(e^{-x})$

$= \frac{1}{2} e^{-x} \sin 2x + \frac{1}{2} \int e^{-x} \sin 2x \, dx$

$= \frac{1}{2} e^{-x} \sin 2x + \frac{1}{2} \int e^{-x} \cdot d\left(-\frac{1}{2} \cos 2x\right)$

$= \frac{1}{2} e^{-x} \sin 2x - \frac{1}{4} e^{-x} \cos 2x + \frac{1}{4} \int \cos 2x \cdot d(e^{-x})$

$= \frac{1}{2} e^{-x} \sin 2x - \frac{1}{4} e^{-x} \cos 2x - \frac{1}{4} \int e^{-x} \cos 2x \, dx$

$\frac{5}{4} \int e^{-x} \cos 2x \, dx = \frac{1}{2} e^{-x} \sin 2x - \frac{1}{4} e^{-x} \cos 2x$

$\int e^{-x} \cos 2x \, dx = \underline{\frac{2}{5} e^{-x} \sin 2x - \frac{1}{5} e^{-x} \cos 2x + c}$ (3)

(c) (i) If a is odd then a^3 is odd (1)

(ii) Let a be odd, ie. $a = 2m + 1$, $m \in \mathbb{Z}$

$a^3 = (2m + 1)^3$

$= 8m^3 + 12m^2 + 6m + 1$

$= 2(4m^3 + 6m^2 + 3m) + 1$

$= 2n + 1$, $n = 4m^3 + 6m^2 + 3m \in \mathbb{Z}$ as $m \in \mathbb{Z}$

$\therefore$ if a is odd then a^3 is odd

$\therefore$ By contraposition, If a^3 is even then a is even. <u>Proof</u> (2)

(iii) As p, q are integers, then $3p^2q + q^3$ is an integer

$\therefore p^3 = 2(3p^2q + q^3)$ is even

$\therefore p$ is even (by part i)

Let $p = 2m$, $m \in \mathbb{Z}$

$(2m)^3 = 2[3(2m)^2q + q^3]$

$8m^3 = 24m^2q + 2q^3$

$q^3 = 4m^3 - 12m^2q$

$= 2(2m^3 - 6m^2q)$

$= 2n$, $n = 2m^3 - 6m^2q \in \mathbb{Z}$ as $m, q \in \mathbb{Z}$

$\therefore q^3$ is even

$\therefore q$ is even (by part i) Proof (3)

(d) (i) Just because the paths cross, does not mean that the missiles must pass through the point of intersection at the same time (1)

(ii) Let A be $(3t_1 - 19, 2t_1 - 14, 28 - t_1)$, B be $(2t_2 - 11, 10 - t_2, 2t_2 + 4)$

$x_A = x_B \Rightarrow 3t_1 - 19 = 2t_2 - 11 \Rightarrow 3t_1 - 2t_2 = 8$ (1)

$y_A = y_B \Rightarrow 2t_1 - 14 = 10 - t_2 \Rightarrow 2t_1 + t_2 = 24$ (2)

(2)×2: $4t_1 + 2t_2 = 48$ (3)

(1)+(3): $7t_1 = 56 \Rightarrow t_1 = 8$

sub in (2): $16 + t_2 = 24 \Rightarrow t_2 = 8$

$\therefore$ IF the trajectories cross, the missiles will collide (they reach the same point at the same time)

$t_1 = 8 \Rightarrow A(5, 2, 20)$

$t_2 = 8 \Rightarrow B(5, 2, 20)$

$\therefore$ Missiles collide at $(5, 2, 20)$ at $t = 8$. (3)

Question 14

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(a) (i) $v^2 = 32 + 8x - 4x^2$

$= -4(x^2 - 2x - 8)$

endpoints: $v = 0$

$x^2 - 2x - 8 = 0$

$(x-4)(x+2) = 0$

$x = -2, 4$

amplitude $= \frac{4-(-2)}{2}$

$A = \underline{3}$

$n^2 = 4$

$n = 2$

period $= \frac{2\pi}{2}$

$= \underline{\pi \text{ seconds}}$ (2)

See alternate solution at end

(ii) centre of motion: $x = \frac{-2+4}{2} = 1$, $n = 2$, $A = 3$

$x = 1 + 3\sin(2t + \alpha)$ OR $x = 1 + 3\cos(2t + \alpha)$

$(t = 0, x = 2.5)$

$2.5 = 1 + 3\sin\alpha$

$\sin\alpha = \frac{1}{2}$

$\alpha = \frac{\pi}{6}, \frac{5\pi}{6}$

$\alpha = \frac{\pi}{6}$ (sin is increasing there)

$2.5 = 1 + 3\cos\alpha$

$\cos\alpha = \frac{1}{2}$

$\alpha = \frac{\pi}{3}, \frac{5\pi}{3}$

$\alpha = \frac{5\pi}{3}$ (cos is increasing there) ($\frac{-\pi}{3}$ is better)

$\underline{x = 1 + 3\sin(2t + \frac{\pi}{6})}$ OR $\underline{x = 1 + 3\cos(2t - \frac{\pi}{3})}$ (2)

(iii) $x = 0$: $0 = 1 + 3\sin(2t + \frac{\pi}{6})$ OR $0 = 1 + 3\cos(2t - \frac{\pi}{3})$

$\sin(2t + \frac{\pi}{6}) = -\frac{1}{3}$

$2t + \frac{\pi}{6} = \pi + \sin^{-1}\frac{1}{3}, 2\pi - \sin^{-1}\frac{1}{3}$

$t = \frac{5\pi}{12} + \frac{1}{2}\sin^{-1}\frac{1}{3}, \frac{11\pi}{12} - \frac{1}{2}\sin^{-1}\frac{1}{3}$

$t = 1.4789, 2.7099$

$\cos(2t - \frac{\pi}{3}) = -\frac{1}{3}$

$2t - \frac{\pi}{3} = -\pi + \cos^{-1}\frac{1}{3}, \pi \pm \cos^{-1}\frac{1}{3}$

$t = -\frac{\pi}{3} + \frac{1}{2}\cos^{-1}\frac{1}{3}, \frac{2\pi}{3} \pm \frac{1}{2}\cos^{-1}(\frac{1}{3})$

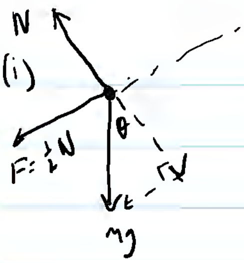
$t = -0.432, 1.4789, 2.7099$

Increasing in speed when moving towards centre, ie. to the right

ie, 1st quadrant answer for sine, 2nd quadrant answer for cos.

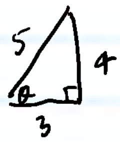
In either case: $\underline{t = 1.4789 \text{ seconds}}$ (3)

(b) (i)



Perpendicular to plane: $N - mg\cos\theta = 0$

$$N = mg\cos\theta$$
$$N = mg \cdot \frac{3}{5}$$



$\cos\theta = \frac{3}{5}$

$$F = 0.5N$$
$$= \frac{3mg}{10}$$

Show (2)

(ii) let x be a distance measured up the plane

$$m\ddot{x} = -F - mg\sin\theta$$
$$= -\frac{3mg}{10} - mg \cdot \frac{4}{5}$$
$$= \frac{-11mg}{10}$$
$$\ddot{x} = -\frac{11g}{10}$$
$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -\frac{11g}{10}$$
$$\left[\frac{1}{2}v^2\right]_V^0 = -\int_0^6 \frac{11g}{10}\,dx$$
$$\left[\frac{1}{2}v^2\right]_0^V = \left[\frac{11g}{10}x\right]_0^6$$
$$\frac{1}{2}V^2 = 6.6g$$
$$V = 11.4 \text{ m/s} \quad (\text{using } g = 9.8)$$

(3)

(c) (i) Every set of three neighbouring integers adds to less than 45. (1)

(ii) $x_1 + x_2 + x_3 < 45$

$x_{30} + x_1 + x_2 < 45$

(1)

(iii) $x_1 + x_2 + x_3 < 45$

$x_2 + x_3 + x_4 < 45$

$\vdots$

$x_{28} + x_{29} + x_{30} < 45$

$x_{29} + x_{30} + x_1 < 45$

$x_{30} + x_1 + x_2 < 45$

Adding: $3(x_1 + x_2 + \dots + x_{30}) < 30 \times 45$

$$x_1 + x_2 + \dots + x_{30} < 450$$

But $x_1 + x_2 + \dots + x_{30} = 1 + 2 + \dots + 30$

$$= \frac{30}{2}(1 + 30)$$
$$= 465$$

contradiction

$\therefore$ statement P is true.

(3)

Question 15

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(a) (i) $\int_a^b f(a+b-x)\,dx$ let $u = a+b-x$

$= \int_b^a f(u)\cdot(-du)$ $du = -dx$

$= \int_a^b f(u)\,du$ $x = a \Rightarrow u = b$

$= \int_a^b f(x)\,dx$ (dummy variable) $x = b \Rightarrow u = a$ Show (1)

(ii) $\int_{\frac{\pi}{6}}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}} = \int_{\frac{\pi}{6}}^{\pi/3} \frac{dx}{1+\sqrt{\tan\left(\frac{\pi}{6}+\frac{\pi}{3}-x\right)}}$ by part (i)

$= \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan\left(\frac{\pi}{2}-x\right)}}$

$= \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\cot x}} \times \frac{\sqrt{\tan x}}{\sqrt{\tan x}}$

$= \int_{\pi/6}^{\pi/3} \frac{\sqrt{\tan x}\,dx}{1+\sqrt{\tan x}}$

$\therefore \int_{\pi/3}^{\pi/6} \frac{dx}{1+\sqrt{\tan x}} = \frac{1}{2}\int_{\pi/6}^{\pi/3}\left(\frac{1}{1+\sqrt{\tan x}} + \frac{\sqrt{\tan x}}{1+\sqrt{\tan x}}\right)dx$

$= \frac{1}{2}\int_{\pi/6}^{\pi/3} dx$

$= \frac{1}{2}\left[x\right]_{\pi/6}^{\pi/3}$

$= \frac{1}{2}\left(\frac{\pi}{3}-\frac{\pi}{6}\right)$

$= \frac{\pi}{12}$ (3)

(b) $P(z) = az^2 + ibz + c$

As w is a zero, $aw^2 + ibw + c = 0$

$P(-\bar{w}) = a(-\bar{w})^2 + ib(-\bar{w}) + c$

$= a\bar{w}^2 - ib\bar{w} + c$

$= \bar{a}\bar{w}^2 + \bar{i}\,\bar{b}\bar{w} + \bar{c}$ (as a, b, c are real and $\bar{z} = z$ if z is real)

$= \bar{a}\,\overline{w^2} + \bar{i}\,\overline{bw} + \bar{c}$ (as $\overline{z^n} = \bar{z}^n$ and $\overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2$)

$= \overline{aw^2} + \overline{ibw} + \bar{c}$ (as $\overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2$)

$= \overline{aw^2 + ibw + c}$ (as $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$)

$= \bar{0}$ (as w is a zero of $P(z)$)

$= 0$

$\therefore -\bar{w}$ is also a root of $P(z) = 0$ Show (2)

(c) (i) let $\underset{\sim}{u} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$, $\underset{\sim}{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$a+b+c = 12 \iff \underset{\sim}{u} \cdot \underset{\sim}{v} = 12$

but $\underset{\sim}{u} \cdot \underset{\sim}{v} = |\underset{\sim}{u}||\underset{\sim}{v}| \cos\theta$

$\leq |\underset{\sim}{u}| \cdot |\underset{\sim}{v}|$ as $\cos\theta \leq 1$

$\therefore |\underset{\sim}{u}||\underset{\sim}{v}| \geq 12$

$|\underset{\sim}{u}|^2 |\underset{\sim}{v}|^2 \geq 144$

$3(a^2+b^2+c^2) \geq 144$

$a^2+b^2+c^2 \geq 48$ Proof (2)

(ii) Equality occurs when $\cos\theta = 1$, ie $\theta = 0, 180°$ ie. vectors are parallel or anti-parallel

ie. $\begin{pmatrix} a \\ b \\ c \end{pmatrix} = k \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow a = k, b = k, c = k$

but $a+b+c = 12 \Rightarrow a = 4, b = 4, c = 4$ Show (1)

(d) (i) $I_k = \int_0^1 x^k (1-x)^{n-k} \, dx$

$= \int_0^1 (1-x)^{n-k} \cdot d\left(\frac{x^{k+1}}{k+1}\right)$

$= \frac{1}{k+1}\left[x^{k+1}(1-x)^{n-k}\right]_0^1 - \frac{1}{k+1}\int_0^1 x^{k+1} \cdot d(1-x)^{n-k}$

$= \frac{1}{k+1}(0-0) - \frac{1}{k+1}\int_0^1 x^{k+1} \cdot (n-k)(1-x)^{n-k-1} \cdot (-1) \, dx$

$= \frac{n-k}{k+1}\int_0^1 x^{k+1}(1-x)^{n-(k+1)} \, dx$

$I_k = \frac{n-k}{k+1} I_{k+1}$

$I_{k+1} = \frac{k+1}{n-k} I_k$

$I_k = \frac{k}{n-(k-1)} I_{k-1} = \frac{k}{n-k+1} I_{k-1}$

OR

$I_k = \int_0^1 x^k \cdot d\left[-\frac{(1-x)^{n-k+1}}{n-k+1}\right]$

$= \frac{1}{n-k+1}\left[x^k(1-x)^{n-k+1}\right]_1^0 + \frac{1}{n-k+1}\int_0^1 (1-x)^{n-k+1} \cdot d(x^k)$

$= \frac{1}{n-k+1}(0-0) + \frac{1}{n-k+1}\int_0^1 (1-x)^{n-k+1} \cdot k x^{k-1} \, dx$

$= 0 + \frac{k}{n-k+1}\int_0^1 x^{k-1}(1-x)^{n-(k-1)} \, dx$

$I_k = \frac{k}{n-k+1} I_{k-1}$

(ii) Let $J_k = \int_0^1 \binom{n}{k} x^k (1-x)^{n-k}\, dx$

$= \binom{n}{k} I_k$

Similarly, $J_{k-1} = \int_0^1 \binom{n}{k-1} x^k (1-x)^{n-k+1}\, dx$

$= \binom{n}{k-1} I_{k-1}$

$$\frac{J_k}{J_{k-1}} = \frac{\binom{n}{k}}{\binom{n}{k-1}} \cdot \frac{I_k}{I_{k-1}}$$

$$= \frac{\cancel{n!}}{k!(n-k)!} \cdot \frac{(k-1)!(n-k+1)!}{\cancel{n!}} \cdot \frac{k}{n-k+1} \quad \text{(from part ii)}$$

$$= \frac{n-k+1}{k} \cdot \frac{k}{n-k+1}$$

$$= 1$$

$\therefore J_k = J_{k-1} \quad \forall k$

$I_0 = \int_0^1 (1-x)^n\, dx$

$$= \frac{1}{n+1}\left[(1-x)^{n+1}\right]_1^0$$

$$= \frac{1}{n+1}(1-0)$$

$$= \frac{1}{n+1}$$

$\therefore J_0 = \binom{n}{0} \cdot \frac{1}{n+1}$

$= \frac{1}{n+1}$

$\therefore J_k = \frac{1}{n+1} \quad \forall k$

Question 16

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(a) Test \$14 : $14 = 8 + 2(3)$

$\therefore$ true for \$14

Assume true for \$n, where $n \geq 14$

Prove true for \$(n+1) :

Case I : There is at least one \$8 coin

Replace one \$8 coin by three \$3 coins,

\$8 has become \$9, so \$n has become \$(n+1).

Case II : There are no \$8 coins.

As n is at least 14, there are at least five \$3 coins.

Replace five \$3 coins by two \$8 coins.

\$15 has become \$16, so \$n has become \$(n+1).

$\therefore$ true for $n+1$ when true for n.

As true for $n=14$, and true for $n+1$ when true for n,

$\therefore$ by MI, every whole dollar amount of \$14 or more can be achieved by taking a mix of \$3 and \$8 coins. Proof (3)

(b) (i) $\frac{2k}{2k+1} - \frac{2k-1}{2k} = \frac{4k^2 - (2k+1)(2k-1)}{2k(2k+1)}$

$= \frac{1}{2k(2k+1)}$

> 0 as $k>0$, so $2k>0$ and $2k+1>0$

$\therefore \frac{2k-1}{2k} < \frac{2k}{2k+1}$ Proof (1)

(ii) From (i), $\frac{1}{2} < \frac{2}{3}$, $\frac{3}{4} < \frac{4}{5}$, $\frac{5}{6} < \frac{6}{7}$, ..., $\frac{2n-1}{2n} < \frac{2n}{2n+1}$

Multiplying : $p < q$

$\therefore p^2 < pq$ (as $p>0$)

$p^2 < \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{\cancel{3}} \cdot \frac{\cancel{3}}{\cancel{4}} \cdot \frac{\cancel{4}}{\cancel{5}} \cdot \frac{\cancel{5}}{\cancel{6}} \cdot \frac{\cancel{6}}{\cancel{7}} \cdots \frac{\cancel{2n-1}}{\cancel{2n}} \cdot \frac{\cancel{2n}}{2n+1}$

$= \frac{1}{2n+1}$

$< \frac{1}{2n}$

$p < \frac{1}{\sqrt{2n}}$ (3) Proof

(c) (i) (I) $\overrightarrow{OP} = \alpha \underset{\sim}{a}$ (II) $\overrightarrow{OP} = \overrightarrow{OB} + \overrightarrow{BP}$

$= \underset{\sim}{b} + \beta \underset{\sim}{c}$

$\therefore \alpha \underset{\sim}{a} = \underset{\sim}{b} + \beta \underset{\sim}{c}$

rearranging: $\underset{\sim}{c} = \frac{\alpha}{\beta} \underset{\sim}{a} - \frac{1}{\beta} \underset{\sim}{b}$

As $\underset{\sim}{a}, \underset{\sim}{b}$ are non-parallel and non-zero :

$\lambda \underset{\sim}{a} + \mu \underset{\sim}{b} = \frac{\alpha}{\beta} \underset{\sim}{a} - \frac{1}{\beta} \underset{\sim}{b} \Rightarrow \lambda = \frac{\alpha}{\beta}, \quad \mu = -\frac{1}{\beta}$

$\frac{\lambda}{\mu} = -\alpha$

$\alpha = -\frac{\lambda}{\mu}$ Show (2)

(ii) $\overrightarrow{QT} = \overrightarrow{RT} - \overrightarrow{RQ}$

$= \phi \overrightarrow{RS} - \overrightarrow{RQ}$

$= \phi \cdot \frac{\overrightarrow{RP} + \overrightarrow{RQ}}{2} - \overrightarrow{RQ}$

$= \frac{\phi}{2} \overrightarrow{RP} + \left(\frac{\phi}{2} - 1\right) \overrightarrow{RQ}$

Similarly, $\overrightarrow{PT} = \frac{\phi}{2} \overrightarrow{RQ} + \left(\frac{\phi}{2} - 1\right) \overrightarrow{RP}$

Applying part (i) where $\overrightarrow{RP} = \underset{\sim}{a}$ and $\overrightarrow{RQ} = \underset{\sim}{b}$:

$\alpha = -\frac{\phi/2}{\phi/2 - 1}$

$= -\frac{\phi}{\phi - 2}$

$\therefore \overrightarrow{RV} = -\frac{\phi}{\phi - 2} \overrightarrow{RP}$

$\overrightarrow{UV} = \overrightarrow{UR} + \overrightarrow{RV}$

$= \frac{\phi}{\phi - 2} \overrightarrow{RP} - \frac{\phi}{\phi - 2} \overrightarrow{RQ}$

$= \frac{\phi}{\phi - 2} (\overrightarrow{RP} - \overrightarrow{RQ})$

$= \frac{\phi}{\phi - 2} \overrightarrow{QP}$

$\therefore \overrightarrow{UV} \parallel \overrightarrow{QP}$ ($\overrightarrow{UV}$ is a scalar multiple of $\overrightarrow{QP}$) [3]

(d) Interpretation of 2nd line: The remainders on dividing m and n by 3 are different.

$$f_0(x) = \frac{1}{1-x}$$

$$f_1(x) = f_0(f_0(x))$$

$$= \frac{1}{1-\frac{1}{1-x}} \times \frac{1-x}{1-x}$$

$$= \frac{1-x}{(1-x)-1}$$

$$= \frac{x-1}{x}$$

$$f_2(x) = f_0(f_1(x))$$

$$= \frac{1}{1-\frac{x-1}{x}} \times \frac{x}{x}$$

$$= \frac{x}{x-(x-1)}$$

$$= x$$

$$f_3(x) = f_0(f_2(x))$$

$$= \frac{1}{1-x} = f_0(x)$$

So $f_{k+3}(x) = f_k(x)$

$f_0(x) = f_1(x)$	$f_0(x) = f_2(x)$	$f_1(x) = f_2(x)$
$\frac{1}{1-x} = \frac{x-1}{x}$	$\frac{1}{1-x} = x$	$\frac{x-1}{x} = x$
$-x^2+2x-1 = x$	$1 = x-x^2$	$x-1 = x^2$
$x^2-x+1=0$	$x^2-x+1=0$	$x^2-x+1=0$

but $x^2-x+1=0$ has the roots of $(x+1)(x^2-x+1)=0$ with $x=-1$ excluded

ie. $x^3+1=0$

$x^3=-1$

ie. solutions are the non-real cube roots of -1

$$-1 = \text{cis}(\pi+2k\pi) \quad k \in \mathbb{Z}$$

$$\sqrt[3]{-1} = \text{cis}\,\frac{2k+1}{3}\pi$$

$k=1$ gives $x=-1$

$k=-1,0$ give $x = \text{cis}\left(\frac{-\pi}{3}\right), \text{cis}\,\frac{\pi}{3}$

ie. $x = e^{\pm i\frac{\pi}{3}}$

[3]

12(a) Alternate solution

$$\int \frac{\sqrt{25x^2-4}}{x}\,dx = \int \frac{\sqrt{25-x^2}}{25x^2}\cdot 25x\,dx$$

let $u^2 = 25x^2 - 4 \Rightarrow 25x^2 = u^2+4$

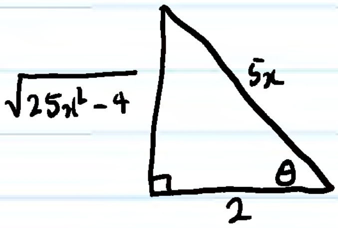
$2u\,du = 50x\,dx$

$25x\,dx = u\,du$

$$= \int \frac{u}{u^2+4}\cdot u\,du$$

$$= \int \frac{u^2\,du}{u^2+4}$$

$$= \int \left(1 - \frac{4}{u^2+4}\right) du$$



$$= u - 2\tan^{-1}\frac{u}{2} + c$$

$$= \sqrt{25x^2-4} - 2\tan^{-1}\frac{\sqrt{25x^2-4}}{2} + c$$

$$= \sqrt{25x^2-4} - 2\cos^{-1}\left(\frac{2}{5x}\right) + c$$

14 a i Alternate Solution

$$v^2 = 32 + 8x - 4x^2$$

$$= -4\left(x^2 - 2x - 8\right)$$

$$= -4\left(9 - (x-1)^2\right) \quad \text{(completing square)}$$

$\therefore$ SHM, centred at $x = 1$, $A^2 = 9 \Rightarrow A = 3$

$n^2 = 4 \Rightarrow \text{per} = \frac{2\pi}{2} = \pi$ seconds